Chapter
2

Trigonometrical Equations and Inequations

Contents

The Greek Mathematician Euclid

The notion of equation is indeed a very old one. The babylonians knew of quadratic equations some 4000 years ago. The Greek Mathematician Euclid (300 B.C.) and Aryabhatta (476 A.D.) gives several Quadratic equations while solving Quadratic and Trigonometrical equations. It was Sridhara an Indian Mathematician around 900 A.D. who was the first to give an algebraic solution of the general equation, and an important treatment by factoring is found in Harriot's work in approximately 1613 A.D., Both these results are very helpful in solving Trigonometrical equations.

Trigonometrical Equations and Inequations

2.1 Introduction

An equation involving one or more trigonometrical ratio of an unknown angle is called a trigonometrical equation *i.e.*, $\sin x + \cos 2x = 1$, $(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$; $\left|\sec\left(\theta + \frac{\pi}{4}\right)\right| = 2$ etc.

A trigonometric equation is different from a trigonometrical identities. An identity is satisfied for every value of the unknown angle *e.g.*, $\cos^2 x = 1 - \sin^2 x$ is true $\forall x \in R$ while a trigonometric equation is satisfied for some particular values of the unknown angle.

(1) **Roots of trigonometrical equation :** The value of unknown angle (a variable quantity) which satisfies the given equation is called the root of an equation *e.g.*, $\cos\theta = \frac{1}{2}$, the root is $\theta = 60^o$ or $\theta = 300^o$ because the equation is satisfied if we put $\theta = 60^o$ or $\theta = 300^o$.

(2) **Solution of trigonometrical equations :** A value of the unknown angle which satisfies the trigonometrical equation is called its solution.

Since all trigonometrical ratios are periodic in nature, generally a trigonometrical equation has more than one solution or an infinite number of solutions. There are basically three types of solutions:

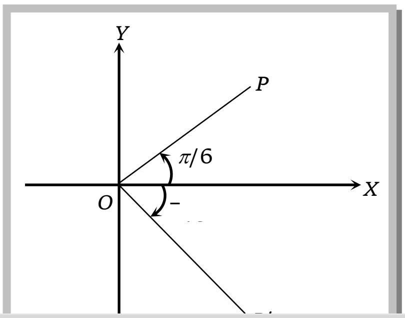
(i) **Particular solution :** A specific value of unknown angle satisfying the equation.

(ii) **Principal solution :** Smallest numerical value of the unknown angle satisfying the equation (Numerically smallest particular solution.)

(iii) **General solution :** Complete set of values of the unknown angle satisfying the equation. It contains all particular solutions as well as principal solutions.

When we have two numerically equal smallest unknown angles, preference is given to the positive value in writing the principal solution. *e.g.*, $\sec\theta = \frac{2}{\sqrt{3}}$ has $\frac{\pi}{6}, -\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{11\pi}{6}, \frac{23\pi}{6}, -\frac{23\pi}{6}$ etc.

As its particular solutions out of these, the numerically smallest are $\frac{\pi}{6}$ and $-\frac{\pi}{6}$ but the principal solution is taken as $\theta = \frac{\pi}{6}$ to write the general solution we notice that the position on



P or P' can be obtained by rotation of OP or OP' around O through a complete angle (2π) any number of times and in any direction (clockwise or anticlockwise)

$\therefore$ The general solution is $\theta = 2k\pi \pm \frac{\pi}{6}, k \in Z$.

2.2 General Solution of Standard Trigonometrical Equations

(1) **General solution of the equation sin θ = sinα:** If $\sin\theta = \sin\alpha$ or $\sin\theta - \sin\alpha = 0$

or, $2\sin\left(\frac{\theta-\alpha}{2}\right)\cos\left(\frac{\theta+\alpha}{2}\right) = 0 \quad \Rightarrow \quad \sin\left(\frac{\theta-\alpha}{2}\right) = 0$ or $\cos\left(\frac{\theta+\alpha}{2}\right) = 0$

or, $\frac{\theta-\alpha}{2} = m\pi; m \in I$ or $\frac{\theta+\alpha}{2} = (2m+1)\frac{\pi}{2}; m \in I$

$\Rightarrow \theta = 2m\pi + \alpha; m \in I$ or $\theta = (2m+1)\pi - \alpha; m \in I$

$\Rightarrow \theta$ = (any even multiple of π) + α or θ = (any odd multiple of π) – α

$$\boxed{\theta = n\pi + (-1)^n\alpha \; ; n \in I}$$

Note : ❑ The equation $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$ is equivalent to $\sin\theta = \sin\alpha$. So these two equation having the same general solution.

(2) **General solution of the equation cosθ = cosα :** If $\cos\theta = \cos\alpha \Rightarrow \cos\theta - \cos\alpha = 0 \Rightarrow$ $-2\sin\left(\frac{\theta+\alpha}{2}\right).\sin\left(\frac{\theta-\alpha}{2}\right) = 0 \Rightarrow \sin\left(\frac{\theta+\alpha}{2}\right) = 0$ or $\sin\left(\frac{\theta-\alpha}{2}\right) = 0$, $\Rightarrow \frac{\theta+\alpha}{2} = n\pi; n \in I$ or $\frac{\theta-\alpha}{2} = n\pi; n \in I$

$\Rightarrow \theta = 2n\pi - \alpha; n \in I$ or $\theta = 2n\pi + \alpha; n \in I$. for the general solution of $\cos\theta = \cos\alpha$, combine these two result which gives $\boxed{\theta = 2n\pi \pm \alpha;\, n \in I}$

Note : ❑ The equation $\sec\theta = \sec\alpha$ is equivalent to $\cos\theta = \cos\alpha$, so the general solution of these two equations are same.

(3) **General solution of the equation tan θ = tan α :** If $\tan\theta = \tan\alpha \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

$\Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0 \Rightarrow \sin(\theta-\alpha) = 0 \Rightarrow \theta - \alpha = n\pi; n \in I$ $\boxed{\theta = n\pi + \alpha; n \in I}$

Note : ❑ The equation $\cot\theta = \cot\alpha$ is equivalent to $\tan\theta = \tan\alpha$ so these two equations having the same general solution.

2.3 General Solution of Some Particular Equations

(1) $\sin\theta = 0 \Rightarrow \theta = n\pi$, $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}$ or $n\pi + \frac{\pi}{2}$, $\tan\theta = 0 \Rightarrow \theta = n\pi$

(2) $\sin\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{2}$ or $2n\pi + \frac{\pi}{2}$, $\cos\theta = 1 \Rightarrow \theta = 2n\pi$, $\tan\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{4}$ or $n\pi + \frac{\pi}{4}$

(3) $\sin\theta = -1 \Rightarrow \theta = (4n+3)\frac{\pi}{2}$ or $2n\pi + \frac{3\pi}{2}$, $\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi$, $\tan\theta = -1 \Rightarrow \theta = (4n-1)\frac{\pi}{4}$ or $n\pi - \frac{\pi}{4}$

(4) $\tan\theta$ = not defined $\Rightarrow \theta = (2n+1)\frac{\pi}{2}$, $\cot\theta$ = not defined $\Rightarrow \theta = n\pi$

cosec θ = not defined $\Rightarrow$ $\theta = n\pi$, sec θ = not defined $\Rightarrow$ $\theta = (2n+1)\frac{\pi}{2}$.

Important Tips

☞ *For equations involving two multiple angles, use multiple and sub-multiple angle formulas, if necessary.*

☞ *For equations involving more than two multiple angles (i) Apply $C \pm D$ formula to combine the two.(ii) Choose such pairs of multiple angle so that after applying the above formulae we get a common factor in the equation.*

Example: 1 If $\sin\theta = \frac{\sqrt{3}}{2}$, then the general value of θ is **[MP PET 1988]**

(a) $2n\pi \pm \frac{\pi}{6}$ (b) $2n\pi \pm \frac{\pi}{3}$ (c) $n\pi + (-1)^n \frac{\pi}{3}$ (d) $n\pi + (-1)^n \frac{\pi}{6}$

Solution: (c) $\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{3} \Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{3}$.

Example: 2 The general solution of $\tan 3x = 1$ is **[Karnataka CET 1991]**

(a) $n\pi + \frac{\pi}{4}$ (b) $\frac{n\pi}{3} + \frac{\pi}{12}$ (c) $n\pi$ (d) $n\pi \pm \frac{\pi}{4}$

Solution: (b) $\tan 3x = \tan\frac{\pi}{4} \Rightarrow 3x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{3} + \frac{\pi}{12}$.

Example: 3 If $\sin 3\theta = \sin\theta$, then the general value of θ is

(a) $2n\pi, (2n+1)\frac{\pi}{3}$ (b) $n\pi, (2n+1)\frac{\pi}{4}$ (c) $n\pi, (2n+1)\frac{\pi}{3}$ (d) None of these

Solution: (b) $\sin 3\theta = \sin\theta$ or $3\theta = m\pi + (-1)^m \theta$

For (m) even *i.e.,* $m = 2n$ then $\theta = \frac{2n\pi}{2} = n\pi$

And for (m) odd, *i.e.,* $m = (2n+1)$ then $\theta = (2n+1)\frac{\pi}{4}$.

Example: 4 The general solution of $2\sin^2\theta - 3\sin\theta - 2 = 0$ is **[Roorkee 1993]**

(a) $n\pi + (-1)^n \frac{\pi}{2}$ (b) $n\pi + (-1)^n \frac{\pi}{6}$ (c) $n\pi + (-1)^n \frac{7\pi}{6}$ (d) $n\pi - (-1)^n \frac{\pi}{6}$

Solution: (d) $2\sin^2\theta - 3\sin\theta - 2 = 0 \Rightarrow 2\sin^2\theta - 4\sin\theta + \sin\theta - 2 = 0 \Rightarrow 2\sin\theta(\sin\theta - 2) + (\sin\theta - 2) = 0$

$(2\sin\theta + 1)(\sin\theta - 2) = 0$

$\sin\theta = +2$ (which is impossible) $\Rightarrow$ $\therefore \sin\theta = -\frac{1}{2}$ $\Rightarrow$ $\sin\theta = \sin(-\pi/6) \Rightarrow \theta = n\pi - (-1)^n \pi/6$.

Example: 5 The number of solutions of the equation $3\sin^2 x - 7\sin x + 2 = 0$ in the interval $[0, 5\pi]$ is **[MP PET 2001; IIT 1998]**

(a) 0 (b) 5 (c) 6 (d) 10

Solution: (c) $3\sin^2 x - 7\sin x + 2 = 0 \Rightarrow 3\sin^2 x - 6\sin x - \sin x + 2 = 0 \Rightarrow (3\sin x - 1)(\sin x - 2) = 0$,

But $\sin x \neq 2$ so $\sin x = \frac{1}{3}$. Hence from 0 to $2\pi = 2$ solution's (one in I[st] quadrant and other in 2[nd] quadrant), from 2π to $4\pi = 2$ solution's and 4π to $5\pi = 2$ solution's. So total number of solutions = 6.

Example: 6 Number of solutions of the equation $\tan x + \sec x = 2\cos x$, lying in the interval $[0, 2\pi]$ is

[AIEEE 2002; MP PET 2000; IIT 1993]

(a) 0 (b) 1 (c) 2 (d) 3

Solution: (c) $\frac{\sin x}{\cos x}+\frac{1}{\cos x}=2\cos x \Rightarrow \sin x+1=2\cos^2 x \Rightarrow 2\sin^2 x+\sin x-1=0 \Rightarrow [2\sin x-1][\sin x+1]=0$

So, $\sin x=-1$ or $\sin x=\frac{1}{2} \Rightarrow x=\frac{3\pi}{2}$ or $x=\frac{\pi}{6},\frac{5\pi}{6}$ but $\frac{3\pi}{2}$ does not satisfy the equation, So total number of solutions $=2$.

Example: 7 If $\tan\theta+\tan 2\theta+\sqrt{3}\tan\theta\tan 2\theta=\sqrt{3}$, then **[UPSEAT 2001]**

(a) $\theta=\frac{(6n+1)\pi}{18}, \forall n\in I$ (b) $\theta=\frac{(6n+1)\pi}{9}, \forall n\in I$ (c) $\theta=\frac{(3n+1)\pi}{9}, \forall n\in I$ (d) None of these

Solution: (c) $\tan\theta+\tan 2\theta+\sqrt{3}\tan\theta\tan 2\theta=\sqrt{3}$ $\Rightarrow \tan\theta+\tan 2\theta=\sqrt{3}(1-\tan\theta\tan 2\theta)$

$\Rightarrow \frac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta}=\sqrt{3} \Rightarrow \tan 3\theta=\tan\left(\frac{\pi}{3}\right) \Rightarrow 3\theta=n\pi+\frac{\pi}{3} \Rightarrow \theta=\frac{n\pi}{3}+\frac{\pi}{9}=(3n+1)\frac{\pi}{9}$.

Example: 8 The solution of the equation $\cos^2 x-2\cos x=4\sin x-\sin 2x, (0\le x\le\pi)$ is **[DCE 2001]**

(a) $\pi-\cot^{-1}\left(\frac{1}{2}\right)$ (b) $\pi-\tan^{-1}(2)$ (c) $\pi+\tan^{-1}\left(-\frac{1}{2}\right)$ (d) None of these

Solution: (c) Given equation is $\cos^2 x-2\cos x=4\sin x-\sin 2x$

$\Rightarrow \cos^2 x-2\cos x=4\sin x-2\sin x\cos x$ $\Rightarrow \cos x(\cos x-2)=2\sin x(2-\cos x)$ $\Rightarrow (\cos x-2)(\cos x+2\sin x)=0$

$\Rightarrow \cos x+2\sin x=0$ $(\because \cos x\ne 2)$ $\Rightarrow \tan x=-\frac{1}{2} \Rightarrow x=n\pi+\tan^{-1}\left(-\frac{1}{2}\right),\ n\in I$

As $0\le x\le\pi$, therefore $x=\pi+\tan^{-1}\left(-\frac{1}{2}\right)$.

Example: 9 The solution of the equation $\begin{vmatrix}\cos\theta & \sin\theta & \cos\theta\\ -\sin\theta & \cos\theta & \sin\theta\\ -\cos\theta & -\sin\theta & \cos\theta\end{vmatrix}=0$, is

(a) $\theta=n\pi$ (b) $\theta=2n\pi\pm\frac{\pi}{2}$ (c) $\theta=n\pi\pm(-1)^n\frac{\pi}{4}$ (d) $\theta=2n\pi\pm\frac{\pi}{4}$

Solution: (b) After solving the determinant $2\cos\theta=0 \Rightarrow \theta=2n\pi\pm\frac{\pi}{2}$.

Example: 10 The general value of θ in the equation $2\sqrt{3}\cos\theta=\tan\theta$, is

(a) $2n\pi\pm\frac{\pi}{6}$ (b) $2n\pi\pm\frac{\pi}{4}$ (c) $n\pi+(-1)^n\frac{\pi}{3}$ (d) $n\pi+(-1)^n\frac{\pi}{4}$

Solution: (c) $2\sqrt{3}\cos^2\theta=\sin\theta \Rightarrow 2\sqrt{3}\sin^2\theta+\sin\theta-2\sqrt{3}=0$

$\Rightarrow \sin\theta=\frac{-1\pm 7}{4\sqrt{3}} \Rightarrow \sin\theta=\frac{-8}{4\sqrt{3}}$ (impossible) or $\sin\theta=\frac{6}{4\sqrt{3}}=\frac{\sqrt{3}}{2} \Rightarrow \theta=n\pi+(-1)^n\frac{\pi}{3}$.

Example: 11 The general value of θ is obtained from the equation $\cos 2\theta=\sin\alpha$ is **[MP PET 1996]**

(a) $2\theta=\frac{\pi}{2}-\alpha$ (b) $\theta=2n\pi\pm\left(\frac{\pi}{2}-\alpha\right)$ (c) $\theta=\frac{n\pi+(-1)^n\alpha}{2}$ (d) $\theta=n\pi\pm\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

Solution: (d) $\cos 2\theta = \sin\alpha \Rightarrow \cos 2\theta = \cos\left(\frac{\pi}{2}-\alpha\right)$

$\therefore\ 2\theta = 2n\pi \pm \left(\frac{\pi}{2}-\alpha\right) \Rightarrow \theta = n\pi \pm \left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

Example: 12 If $\begin{vmatrix} \cos(A+B) & -\sin(A+B) & \cos 2B \\ \sin A & \cos A & \sin B \\ -\cos A & \sin A & \cos B \end{vmatrix} = 0$, then $B =$ **[EAMCET 2003]**

(a) $(2n+1)\frac{\pi}{2}$ (b) $n\pi$ (c) $(2n+1)\pi$ (d) $2n\pi$

Solution: (a) On expanding the determinant $\cos^2(A+B)+\sin^2(A+B)+\cos 2B = 0$

$1+\cos 2B = 0$ or $\cos 2B = \cos\pi$ or $2B = 2n\pi+\pi$ or $B = (2n+1)\frac{\pi}{2}$.

Example: 13 If $\cos\theta+\cos 2\theta+\cos 3\theta = 0$, then the general value of θ is

(a) $\theta = 2m\pi \pm \frac{2\pi}{3}$ (b) $\theta = 2m\pi \pm \frac{\pi}{4}$ (c) $\theta = m\pi+(-1)^m\frac{2\pi}{3}$ (d) $\theta = m\pi+(-1)^m\frac{\pi}{3}$

Solution: (a) $\Rightarrow \cos\theta+\cos 2\theta+\cos 3\theta = 0 \Rightarrow (\cos\theta+\cos 3\theta)+\cos 2\theta = 0$

$\Rightarrow 2\cos 2\theta.\cos\theta+\cos 2\theta = 0 \Rightarrow \cos 2\theta(2\cos\theta+1) = 0$

$\Rightarrow \cos 2\theta = 0 = \cos\frac{\pi}{2} \Rightarrow 2\theta = 2m\pi \pm \pi/2 \Rightarrow \theta = m\pi \pm \frac{\pi}{4}$ or $\cos\theta = -\frac{1}{2} = \cos\frac{2\pi}{3} \Rightarrow \theta = 2m\pi \pm \frac{2\pi}{3}$.

Example: 14 $\sin 6\theta+\sin 4\theta+\sin 2\theta = 0$, then θ equal to

(a) $\frac{n\pi}{4}$ or $n\pi \pm \frac{\pi}{3}$ (b) $\frac{n\pi}{4}$ or $n\pi \pm \frac{\pi}{6}$ (c) $\frac{n\pi}{4}$ or $2n\pi \pm \frac{\pi}{6}$ (d) None of these

Solution: (a) $(\sin 6\theta+\sin 2\theta)+\sin 4\theta = 0 \Rightarrow 2\sin 4\theta\cos 2\theta+\sin 4\theta = 0 \Rightarrow \sin 4\theta(2\cos 2\theta+1) = 0 \Rightarrow \sin 4\theta = 0$ or $2\cos 2\theta+1 = 0$

$\sin 4\theta = \sin 0$ or $\cos 2\theta = \cos\frac{2\pi}{3}$

$4\theta = n\pi$ or $2\theta = 2n\pi \pm \frac{2\pi}{3}$

$\theta = \frac{n\pi}{4}$ or $\theta = n\pi \pm \frac{\pi}{3}$.

2.4 General Solution of Square of Trigonometrical Equations

(1) **General solution of $\sin^2\theta = \sin^2\alpha$:** If $\sin^2\theta = \sin^2\alpha$ or, $2\sin^2\theta = 2\sin^2\alpha$ (Both the sides multiply by 2) or, $1-\cos 2\theta = 1-\cos 2\alpha$ or, $\cos 2\theta = \cos 2\alpha$, $2\theta = 2n\pi \pm 2\alpha;\ n\in I$, $\boxed{\theta = n\pi \pm \alpha;\ n\in I}$

(2) **General solution of $\cos^2\theta = \cos^2\alpha$:** If $\cos^2\theta = \cos^2\alpha$ or, $2\cos^2\theta = 2\cos^2\alpha$ (multiply both the side by 2) or, $1+\cos 2\theta = 1+\cos 2\alpha$ or, $2\theta = 2n\pi \pm 2\alpha$; $\boxed{\theta = n\pi \pm \alpha;\ n\in I}$

(3) **General solution of $\tan^2\theta = \tan^2\alpha$**: If $\tan^2\theta = \tan^2\alpha$ or, $\frac{\tan^2\theta}{1} = \frac{\tan^2\alpha}{1}$

Using componendo and dividendo rule, $\frac{\tan^2\theta+1}{\tan^2\theta-1}=\frac{\tan^2\alpha+1}{\tan^2\alpha-1}$

or $\frac{1+\tan^2\theta}{1-\tan^2\theta}=\frac{1+\tan^2\alpha}{1-\tan^2\alpha}$ or $\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-\tan^2\alpha}{1+\tan^2\alpha}$ or $\cos 2\theta=\cos 2\alpha$, $\boxed{\theta=n\pi\pm\alpha;\, n\in I}$

Example: 15 General value of θ satisfying the equation $\tan^2\theta+\sec 2\theta=1$ is **[IIT 1996]**

(a) $m\pi, n\pi+\frac{\pi}{3}$ (b) $m\pi, n\pi\pm\frac{\pi}{3}$ (c) $m\pi, n\pi\pm\frac{\pi}{6}$ (d) None of these

Solution: (b) $\tan^2\theta+\sec 2\theta=1 \Rightarrow \tan^2\theta+\frac{1+\tan^2\theta}{1-\tan^2\theta}=1 \Rightarrow \tan^2\theta-\tan^4\theta+1+\tan^2\theta=1-\tan^2\theta$

$\tan^4\theta-3\tan^2\theta=0 \Rightarrow \tan^2\theta(\tan^2\theta-3)=0 \Rightarrow \tan^2\theta=0$ and $\tan^2\theta=3$

$\tan^2\theta=\tan^2 0$ and $\tan^2\theta=\tan^2\frac{\pi}{3} \Rightarrow \theta=m\pi$ and $\theta=n\pi\pm\frac{\pi}{3}$.

Example: 16 If $\sec^2\theta=\frac{4}{3}$, then the general value of θ is **[MP PET 1988]**

(a) $2n\pi\pm\frac{\pi}{6}$ (b) $n\pi\pm\frac{\pi}{6}$ (c) $2n\pi\pm\frac{\pi}{3}$ (d) $n\pi\pm\frac{\pi}{3}$

Solution: (b) $\sec^2\theta=\frac{4}{3} \Rightarrow \cos^2\theta=\frac{3}{4}=\left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \cos^2\theta=\cos^2\frac{\pi}{6}$

$\therefore \theta=n\pi\pm\frac{\pi}{6}$.

Example: 17 If $2\tan^2\theta=\sec^2\theta$, then the general value of θ is **[MP PET 1989]**

(a) $n\pi+\frac{\pi}{4}$ (b) $n\pi-\frac{\pi}{4}$ (c) $n\pi\pm\frac{\pi}{4}$ (d) $2n\pi\pm\frac{\pi}{4}$

Solution: (c) $2\tan^2\theta=1+\tan^2\theta \Rightarrow \tan^2\theta=1 \Rightarrow \tan^2\theta=\tan^2\frac{\pi}{4}$

$\therefore \theta=n\pi\pm\frac{\pi}{4}$.

Example: 18 If $\sin^2\theta=\frac{1}{4}$, then the most general value of θ is **[MP PET 1984, 90; UPSEAT 1973]**

(a) $2n\pi\pm(-1)^n\frac{\pi}{6}$ (b) $\frac{n\pi}{2}\pm(-1)^n\frac{\pi}{6}$ (c) $n\pi\pm\frac{\pi}{6}$ (d) $2n\pi\pm\frac{\pi}{6}$

Solution: (c) $\sin^2\theta=\frac{1}{4}=\left(\frac{1}{2}\right)^2 \Rightarrow \sin^2\theta=\sin^2\frac{\pi}{6}$

$\therefore \theta=n\pi\pm\frac{\pi}{6}$.

Example: 19 If $\frac{1-\cos 2\theta}{1+\cos 2\theta}=3$, then the general value of θ is

(a) $2n\pi \pm \frac{\pi}{6}$ (b) $n\pi \pm \frac{\pi}{6}$ (c) $2n\pi + \frac{\pi}{3}$ (d) $n\pi \pm \frac{\pi}{3}$

Solution: (d) $\frac{1-\cos 2\theta}{1+\cos 2\theta} = 3 \Rightarrow \frac{2\sin^2\theta}{2\cos^2\theta} = 3 \Rightarrow \tan^2\theta = 3 = (\sqrt{3})^2 \Rightarrow \tan^2\theta = \tan^2\frac{\pi}{3}$

$\theta = n\pi \pm \frac{\pi}{3}$.

2.5 Solutions in the Case of Two Equations are given (Simultaneously Solving Equation).

We may divide the problem into two categories. (1) Two equations in one 'unknown' satisfied simultaneously. (2) Two equations in two 'unknowns' satisfied simultaneously.

(1) **Two equations is one 'unknown' :** Two equations are given and we have to find the values of variables θ which may satisfy with the given equations.

(i) $\cos\theta = \cos\alpha$ and $\sin\theta = \sin\alpha$, so the common solution is $\boxed{\theta = 2n\pi + \alpha, n \in I}$

(ii) $\sin\theta = \sin\alpha$ and $\tan\theta = \tan\alpha$, so the common solution is $\boxed{\theta = 2n\pi + \alpha, n \in I}$

(iii) $\cos\theta = \cos\alpha$ and $\tan\theta = \tan\alpha$, so the common solution is $\boxed{\theta = 2n\pi + \alpha, n \in I}$

Example: 20 The most general value of θ satisfying the equation $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$ is

[MP PET 2003; UPSEAT 2002, 1982; Roorkee 1990]

(a) $n\pi + \frac{7\pi}{4}$ (b) $n\pi + (-1)^n \frac{7\pi}{4}$ (c) $2n\pi + \frac{7\pi}{4}$ (d) None of these

Solution: (c) $\tan\theta = -1 = \tan\left(2\pi - \frac{\pi}{4}\right)$ and $\cos\theta = \frac{1}{\sqrt{2}} = \cos\left(2\pi - \frac{\pi}{4}\right)$

Hence, general value is $2n\pi + \left(2\pi - \frac{\pi}{4}\right) = 2n\pi + \frac{7\pi}{4}$.

Example: 21 The most general value of θ which will satisfy both the equations $\sin\theta = \frac{-1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ is

[MNR 1980; MP PET 1989; DCE 1995]

(a) $n\pi + (-1)^n \frac{\pi}{6}$ (b) $n\pi + \frac{\pi}{6}$ (c) $2n\pi \pm \frac{\pi}{6}$ (d) None of these

Solution: (d) $\sin\theta = \frac{-1}{2} = \sin\left(\frac{-\pi}{6}\right) = \sin\left(\pi + \frac{\pi}{6}\right)$

$\tan\theta = \left(\frac{1}{\sqrt{3}}\right) = \tan\left(\frac{\pi}{6}\right) = \tan\left(\pi + \frac{\pi}{6}\right) \Rightarrow \theta = \left(\pi + \frac{\pi}{6}\right)$

Hence, general value of θ is $2n\pi + \frac{7\pi}{6}$.

(2) **System of equations (Two equations in two unknowns) :** Let $f(\theta,\phi) = 0, g(\theta,\phi) = 0$ be the system of two equations in two unknowns.

Step (i) **:** Eliminate any one variable, say ϕ. Let $\theta = \alpha$ be one solution.

Step (ii) **:** Then consider the system $f(\alpha,\phi) = 0,\ g(\alpha,\phi) = 0$ and use the method of two equations in one variable.

Note : ❑ It is preferable to solve the system of equations quadrant wise.

Example: 22 If $\left(\frac{\sin\theta}{\sin\phi}\right)^2 = \frac{\tan\theta}{\tan\phi} = 3$, then the value of θ and ϕ are

(a) $\theta = n\pi \pm \frac{\pi}{3}, \phi = n\pi \pm \frac{\pi}{6}$ (b) $\theta = n\pi - \frac{\pi}{3}, \phi = n\pi - \frac{\pi}{6}$

(c) $\theta = n\pi \pm \frac{\pi}{2}, \phi = n\pi + \frac{\pi}{3}$ (d) None of these

Solution: (a) $\left(\frac{\sin\theta}{\sin\phi}\right)^2 = \frac{\tan\theta}{\tan\phi} \Rightarrow \sin\theta\cos\theta = \sin\phi\cos\phi \Rightarrow \sin 2\theta = \sin 2\phi$

$2\theta = \pi - 2\phi \Rightarrow \theta = \frac{\pi}{2} - \phi$

But, $\frac{\tan\theta}{\tan\phi} = 3 \Rightarrow \frac{\tan\theta}{\cot\theta} = 3 \Rightarrow \tan^2\theta = 3 \Rightarrow \theta = n\pi \pm \frac{\pi}{3}$, so that $\phi = n\pi \pm \frac{\pi}{6}$

Trick: Check with the options for $n = 0, n = 1$.

Example: 23 Solve the system of equations $\sec\theta = \sqrt{2}\sec\phi$, $\tan\theta = \sqrt{3}\tan\phi$

Solution: Usually students proceed this type of problems in the following way:

Squaring and subtracting, we get $\sec^2\theta - \tan^2\theta = 2\sec^2\phi - 3\tan^2\phi$,

i.e., $2\tan^2\phi + 2 - 3\tan^2\phi = 1$ or $\tan^2\phi = 1$ or $\phi = n\pi \pm \frac{\pi}{4}$ ....(i)

Also we have $\sec^2\phi - \tan^2\phi = \frac{\sec^2\theta}{2} - \frac{\tan^2\theta}{3}$

which gives $6 = 3\sec^2\theta - 2\tan^2\theta$ or $\tan^2\theta = 3$, and so $\theta = m\pi \pm \frac{\pi}{3}$.

Thus solution of this system is $\theta = m\pi \pm \frac{\pi}{3}$ and $\phi = n\pi \pm \frac{\pi}{4}, m, n \in I$ ....(ii)

Now see the fallacies: $\theta = \frac{\pi}{3}$ and $\phi = -\frac{\pi}{4}$ (from the solution) give $\sec\left(\frac{\pi}{3}\right) = \sqrt{2}\sec\left(-\frac{\pi}{4}\right)$

i.e., $2 = 2$, but $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}\tan\left(-\frac{\pi}{4}\right)$ give $\sqrt{3} = -\sqrt{3}$.

Thus solution given in (ii) consists many extraneous (absurd) solutions. The simple reason for this is quite obvious. (ii) consists of solutions of following four systems:

$\sec\theta = \sqrt{2}\sec\phi$, $\tan\theta = \sqrt{3}\tan\phi$ ....(iii)

$\sec\theta = \sqrt{2}\sec\phi$, $\tan\theta = -\sqrt{3}\tan\phi$ ....(iv)

$\sec\theta = -\sqrt{2}\sec\phi$, $\tan\theta = \sqrt{3}\tan\phi$ ....(v)

and $\sec\theta = -\sqrt{2}\sec\phi$, $\tan\theta = -\sqrt{3}\tan\phi$ ....(vi)

While we have to find the values which satisfy (iii). Therefore, we have to verify the solutions and should retain only the valid ones.

Alternative Method : A better method for such type of equations is following:

The given system is $\sec\theta = \sqrt{2}\sec\phi$ ....(vii)

$\tan\theta = \sqrt{3}\tan\phi$ ....(viii)

$(vii)^2 - (viii)^2$ gives $\tan^2\phi = 1 \quad \therefore \phi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Case 1 : $\phi = \frac{\pi}{4}$, the system reduces to $\sec\theta = 2,\ \tan\theta = \sqrt{3}$, so $\theta = \frac{\pi}{3}$.

$\therefore \theta = 2n\pi + \frac{\pi}{3},\ \phi = 2m\pi + \frac{\pi}{4}$. ....(ix)

Case 2 : $\phi = \frac{3\pi}{4}$, then we have $\sec\theta = -2,\ \tan\theta = -\sqrt{3}$, so $\theta = \frac{2\pi}{3}$.

Thus general solution is $\theta = 2n\pi + \frac{2\pi}{3},\ \phi = 2m\pi + \frac{3\pi}{4}$ ....(x)

Case 3 : $\phi = \frac{5\pi}{4}$ (or can be taken as $-\frac{3\pi}{4}$)

Then $\sec\theta = -2,\ \tan\theta = \sqrt{3},\ \therefore \theta \in Q_3$, so $\theta = \frac{4\pi}{3}$. Thus $\theta = 2n\pi + \frac{4\pi}{3}$, $\phi = 2m\pi + \frac{5\pi}{4}$ or $\theta = 2n\pi - \frac{2\pi}{3}$, $\phi = 2m\pi - \frac{3\pi}{4}$ ....(xi)

Case 4: $\phi = \frac{7\pi}{4}$ (or $-\frac{\pi}{4}$).

Then $\sec\theta = 2, \tan\theta = -\sqrt{3}$, so $\theta = -\frac{\pi}{3}$ and so $\theta = 2n\pi - \frac{\pi}{3},\ \phi = 2m\pi - \frac{\pi}{4}$. ....(xii)

Hence, the required solutions are given as $(\theta,\phi) = \left(2n\pi + \frac{\pi}{3},\ 2m\pi + \frac{\pi}{4}\right);\ \left(2n\pi + \frac{2\pi}{3},\ 2m\pi + \frac{3\pi}{4}\right)$, $\left(2n\pi - \frac{2\pi}{3},\ 2m\pi - \frac{3\pi}{4}\right), \left(2n\pi - \frac{\pi}{3},\ 2m\pi - \frac{\pi}{4}\right)$.

Note : ❑ Do not write the solution as $\theta = 2n\pi + \frac{\pi}{3},\ 2n\pi + \frac{2\pi}{3}$,......... ...., $\phi = 2m\pi + \frac{\pi}{4}$,......... .

2.6 General Solution of the form a cosθ + bsinθ = c

In $a\cos\theta + b\sin\theta = c$, put $a = r\cos\alpha$ and $b = r\sin\alpha$ where $r = \sqrt{a^2+b^2}$ and $|c| \le \sqrt{a^2+b^2}$

Then, $r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = c \ \Rightarrow \cos(\theta-\alpha) = \frac{c}{\sqrt{a^2+b^2}} = \cos\beta$ (say) ....(i)

$\Rightarrow \theta - \alpha = 2n\pi \pm \beta \Rightarrow \theta = 2n\pi \pm \beta + \alpha$, where $\tan\alpha = \frac{b}{a}$, is the general solution

Alternatively, putting $a = r\sin\alpha$ and $b = r\cos\alpha$ where $r = \sqrt{a^2+b^2}$ $\Rightarrow \sin(\theta+\alpha) = \frac{c}{\sqrt{a^2+b^2}} = \sin\gamma$ (say)

$\Rightarrow \theta + \alpha = n\pi + (-1)^n\gamma \ \Rightarrow \theta = n\pi + (-1)^n\gamma - \alpha$, where $\tan\alpha = \frac{a}{b}$, is the general solution.

Note : ❑ $(-\sqrt{a^2+b^2}) \le a\cos\theta + b\sin\theta \le (\sqrt{a^2+b^2})$

❑ The general solution of $a\cos x + b\sin x = c$ is $x = 2n\pi + \tan^{-1}\left(\frac{b}{a}\right) \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)$.

Example: 24 The number of integral values of k, for which the equation $7\cos x + 5\sin x = 2k+1$ has a solution is **[IIT Screeni**

(a) 4 (b) 8 (c) 10 (d) 12

Solution: (b) $-\sqrt{7^2+5^2} \le (7\cos x + 5\sin x) \le \sqrt{7^2+5^2}$

So, for solution $-\sqrt{74} \le (2k+1) \le \sqrt{74}$ or $-8.6 \le 2k+1 \le 8.6$ or $-9.6 \le 2k \le 7.6$ or $-4.8 \le k \le 3.8$.

So, integral values of k are $-4,-3,-2,-1,0,1,2,3$ (eight values)

Example: 25 If $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$, then general value of θ is **[MP PET 2002, 1991; UPSEAT 1999]**

(a) $n\pi + (-1)^n \frac{\pi}{4}$ (b) $(-1)^n \frac{\pi}{4} - \frac{\pi}{3}$ (c) $n\pi + \frac{\pi}{4} - \frac{\pi}{3}$ (d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}$

Solution: (d) $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2} \Rightarrow \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\frac{\pi}{3}\cos\theta + \cos\frac{\pi}{3}\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \sin\left(\theta + \frac{\pi}{3}\right) = \sin\frac{\pi}{4} \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{3}$.

Example: 26 If $\sin\theta + \cos\theta = 1$, then the general value of θ is **[Karnataka CET 2002; DCE 2000; MNR 1987; IIT 1981]**

(a) $2n\pi$ (b) $n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4}$ (c) $2n\pi + \frac{\pi}{2}$ (d) None of these

Solution: $\sin\theta + \cos\theta = 1 \Rightarrow \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$ $\left[\text{Dividing by } \sqrt{1^2+1^2} = \sqrt{2}\right]$

$\Rightarrow \sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \Rightarrow \theta + \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{4} \Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4}$.

Example: 27 The equation $\sqrt{3}\sin x + \cos x = 4$ has **[EAMCET 2001]**

(a) Only one solution (b) Two solutions

(c) Infinitely many solutions (d) No solution

Solution: (d) Given equation is $\sqrt{3}\sin x + \cos x = 4$ which is of the form $a\sin x + b\cos x = c$ with $a = \sqrt{3}, b = 1, c = 4$

Here $a^2 + b^2 = 3 + 1 = 4 < c^2$, Therefore the given equation has no solution.

Example: 28 The general solution of the equation $(\sqrt{3}-1)\sin\theta + (\sqrt{3}+1)\cos\theta = 2$ is **[Roorkee 1992]**

(a) $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ (b) $n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{12}$ (c) $2n\pi \pm \frac{\pi}{4} - \frac{\pi}{12}$ (d) $n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{12}$

Solution: (a) $(\sqrt{3}-1)\sin\theta + (\sqrt{3}+1)\cos\theta = 2$

Divided by $\sqrt{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2} = 2\sqrt{2}$ in both sides,

We get, $\frac{(\sqrt{3}-1)}{2\sqrt{2}}\sin\theta + \frac{(\sqrt{3}+1)}{2\sqrt{2}}\cos\theta = \frac{2}{2\sqrt{2}}$

$\sin\theta\sin 15° + \cos\theta\cos 15° = \frac{1}{\sqrt{2}} \Rightarrow \sin\theta . \sin\frac{\pi}{12} + \cos\theta . \cos\frac{\pi}{12} = \cos\frac{\pi}{4}$

$\cos\left(\theta - \frac{\pi}{12}\right) = \cos\frac{\pi}{4} \Rightarrow \theta - \frac{\pi}{12} = 2n\pi \pm \frac{\pi}{4} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$.

2.7 Some Particular Equations

(1) **Equation of the form $a_0\sin^n x + a_1\sin^{n-1}x\cos x + a_2\sin^{n-2}x\cos^2 x + ... + a_n\cos^n x = 0$:** Here $a_0, a_1 ..., a_n$ are real numbers and the sum of the exponents in $\sin x$ and $\cos x$ in each term is equal to n, are said to be homogeneous with respect to sin$x$ and cos$x$. For $\cos x \ne 0$, above equation can be written as, $a_0\tan^n x + a_1\tan^{n-1}x + ... + a_n = 0$.

Example: 29 The solution of equation $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4$ is

(a) $x = n\pi + \tan^{-1} 3$ or $x = n\pi + \tan^{-1} 4$ (b) $x = n\pi + \frac{\pi}{6}$ or $x = n\pi + \frac{\pi}{4}$

(c) $x = n\pi$ or $x = n\pi + \frac{\pi}{4}$ (d) None of these

Solution: (a) To solve this kind of equation; we use the fundamental formula trigonometrical identity, $\sin^2 x + \cos^2 x = 1$

writing the equation in the form, $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4(\sin^2 x + \cos^2 x)$

$\Rightarrow \sin^2 x - 7\sin x\cos x + 12\cos^2 x = 0$

Dividing by $\cos^2 x$ on both sides we get, $\tan^2 x - 7\tan x + 12 = 0$

Now it can be factorized as; $(\tan x - 3)(\tan x - 4) = 0 \Rightarrow \tan x = 3, 4$

i.e., $\tan x = \tan(\tan^{-1} 3)$ or $\tan x = \tan(\tan^{-1} 4) \Rightarrow x = n\pi + \tan^{-1} 3$ or $x = n\pi + \tan^{-1} 4$.

(2) **A trigonometric equation of the form $R(\sin kx, \cos nx, \tan mx, \cot lx) = 0$** : Here R is a rational function of the indicated arguments and (k, l, m, n are natural numbers) can be reduced to a rational equation with respect to the arguments $\sin x, \cos x, \tan x$, and $\cot x$ by means of the formulae for trigonometric functions of the sum of angles (in particular, the formulas for double and triple angles) and then reduce equation of the given form to a rational equation with respect to the unknown, $t = \tan\frac{x}{2}$ by means of the formulas,

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}, \cot x = \frac{1-\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

Example: 30 If $(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0$ then $x =$

(a) $2n\pi \pm \frac{\pi}{3}$ (b) $n\pi \pm \frac{\pi}{3}$ (c) $2n\pi \pm \frac{\pi}{6}$ (d) None of these

Solution: (a) Let $t = \tan\frac{x}{2}$, and using the formula. We get, $\left\{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} - \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right\}\left\{\frac{4\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} + \frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right\} + 2 = 0$

$$\left(\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}\right)\left(\frac{4t}{1-t^2} + \frac{1+t^2}{1-t^2}\right) + 2 = 0 \Rightarrow \frac{3t^4 + 6t^3 + 8t^2 - 2t - 3}{(t^2+1)(1-t^2)} = 0$$

Its roots are; $t_1 = \frac{1}{\sqrt{3}}$ and $t_2 = -\frac{1}{\sqrt{3}}$.

Thus the solution of the equation reduces to that of two elementary equations,

$\tan\frac{x}{2} = \frac{1}{\sqrt{3}}, \tan\frac{x}{2} = -\frac{1}{\sqrt{3}} \Rightarrow \frac{x}{2} = n\pi \pm \frac{\pi}{6} \Rightarrow x = 2n\pi \pm \frac{\pi}{3}$, is required solution.

(3) **Equation of the form** $R(\sin x + \cos x, \sin x.\cos x) = 0$: where R is rational function of the arguments in brackets, Put $\sin x + \cos x = t$ ....(i) and use the following identity:

$$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x \cos x = 1 + 2\sin x \cos x \Rightarrow \sin x \cos x = \frac{t^2 - 1}{2} \quad ....(ii)$$

Taking (i) and (ii) into account, we can reduce given equation into; $R\left(t, \frac{t^2-1}{2}\right) = 0$.

Similarly, by the substitution $(\sin x - \cos x) = t$, we can reduce the equation of the form; $R(\sin x - \cos x, \sin x \cos x) = 0$ to an equation; $R\left(t, \frac{1-t^2}{2}\right) = 0$.

Example: 31 If $\sin x + \cos x - 2\sqrt{2}\sin x \cos x = 0$ then the general solution of x is

(a) $x = 2n\pi + \frac{\pi}{4}$ (b) $x = n\pi + (-1)^n \frac{\pi}{6} - \frac{\pi}{4}$ (c) Both (a) and (b) (d) None of these

Solution: (c) Let $(\sin x + \cos x) = t$ and using the equation $\sin x . \cos x = \frac{t^2-1}{2}$, we get $t - 2\sqrt{2}\left(\frac{t^2-1}{2}\right) = 0 \Rightarrow$ $\sqrt{2}t^2 - t - \sqrt{2} = 0$

The numbers $t_1 = \sqrt{2}, t_2 = -\frac{1}{\sqrt{2}}$ are roots of this quadratic equation.

Thus the solution of the given equation reduces to the solution of two trigonometrical equation;

$\sin x + \cos x = \sqrt{2}$ or $\sin x + \cos x = -\frac{1}{\sqrt{2}}$

or $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1$ or $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{2}$

or $\sin x . \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = 1$ or $\sin x \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = -\frac{1}{2}$

$\Rightarrow \sin\left(x + \frac{\pi}{4}\right) = 1$ or $\sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{2} \Rightarrow x + \frac{\pi}{4} = (4n+1)\frac{\pi}{2}$ or $x + \frac{\pi}{4} = n\pi + (-1)^n . \left(\frac{\pi}{6}\right)$

$\Rightarrow x = 2n\pi + \frac{\pi}{4}$ or $x = n\pi + (-1)^n \frac{\pi}{6} - \left(\frac{\pi}{4}\right)$.

Example: 32 If $\sin^{10} x + \cos^{10} x = \frac{29}{16}\cos^4 2x$. then $x =$

(a) $x = \frac{n\pi}{4}$ (b) $x = \frac{n\pi}{4} + \frac{\pi}{8}$ (c) $x = \frac{n\pi}{4} + \frac{\pi}{3}$ (d) None of these

Solution: (b) Using half-angle formulae we can represent the given equation in the form, $\left(\frac{1-\cos 2x}{2}\right)^5 + \left(\frac{1+\cos 2x}{2}\right)^5 = \frac{29}{16}\cos^4 2x$

Put $\cos 2x = t$, $\left(\frac{1-t}{2}\right)^5 + \left(\frac{1+t}{2}\right)^5 = \frac{29}{16}t^4 \Rightarrow 24t^4 - 10t^2 - 1 = 0$ whose only real root is, $t^2 = \frac{1}{2}$.

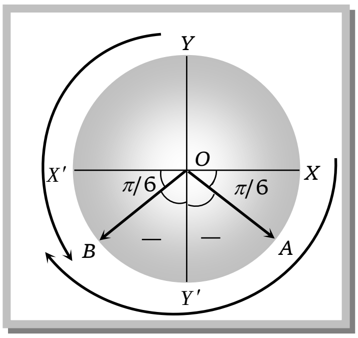
$\therefore \cos^2 2x = \frac{1}{2} \Rightarrow 1 + \cos 4x = 1 \Rightarrow \cos 4x = 0 \Rightarrow 4x = (2n+1)\frac{\pi}{2} \Rightarrow x = \frac{n\pi}{4} + \frac{\pi}{8}$; $n \in I$

Note : ❑ Some trigonometric equations can sometimes be simplified by lowering their degrees. If the exponent of the sines and cosines occuring into an equation are even, the lowering of the degree can be done by half angle formulas as in above example.

2.8 Method for Finding Principal Value

Suppose we have to find the principal value of θ satisfying the equation $\sin\theta = -\frac{1}{2}$.

Since $\sin\theta$ is negative, θ will be in 3rd or 4th quadrant. We can approach 3rd or 4th quadrant from two directions. If we take anticlockwise direction the numerical value of the angle will be greater than π. If we approach it in clockwise direction the angle will be numerically less than π. For principal value, we have to take numerically smallest angle. So for principal value



(1) If the angle is in 1st or 2nd quadrant we must select anticlockwise direction and if the angle is in 3rd or 4th quadrant, we must select clockwise direction.

(2) Principal value is never numerically greater than π.

(3) Principal value always lies in the first circle (*i.e.*, in first rotation). On the above criteria, θ will be $-\frac{\pi}{6}$ or $-\frac{5\pi}{6}$. Among these two $-\frac{\pi}{6}$ has the least numerical value. Hence $-\frac{\pi}{6}$ is the principal value of θ satisfying the equation $\sin\theta = -\frac{1}{2}$.

From the above discussion, the method for finding principal value can be summed up as follows :

(i) First draw a trigonometrical circle and mark the quadrant, in which the angle may lie.

(ii) Select anticlockwise direction for 1st and 2nd quadrants and select clockwise direction for 3rd and 4th quadrants.

(iii) Find the angle in the first rotation.

(iv) Select the numerically least angle. The angle thus found will be principal value.

(v) In case, two angles one with positive sign and the other with negative sign qualify for the numerically least angle, then it is the convention to select the angle with positive sign as principal value.

Example: 33 If $\cos\theta + \sqrt{3}\sin\theta = 2$, then $\theta =$ (only principal value)

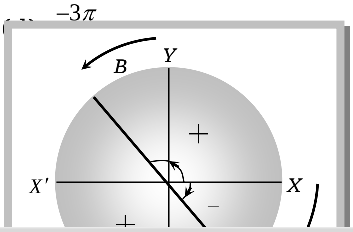
(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{5\pi}{3}$

Solution: (a) $\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta = \frac{2}{2} \Rightarrow \cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3} = 1 \Rightarrow \cos\left(\theta - \frac{\pi}{3}\right) = 1 = \cos 0° \Rightarrow \theta - \frac{\pi}{3} = 0° \Rightarrow \theta = \frac{\pi}{3}$.

Example: 34 Principal value of $\tan\theta = -1$ is

(a) $\frac{-\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{-3\pi}{4}$

Solution: (a) $\because \tan\theta$ is negative.



$\therefore$ θ will lie in 2nd or 4th quadrant. For 2nd quadrant we will select anticlockwise and for 4th quadrant, we will select clockwise direction.

In the first circle two values $\frac{-\pi}{4}$ and $\frac{3\pi}{4}$ are obtained.

Among these two, $\frac{-\pi}{4}$ is numerically least angle. Hence principal value is $\frac{-\pi}{4}$.

Important Tips

☞ *Any trigonometric equation can be solved without using any formula. Find all angles in* $[0, 2\pi]$ *which satisfy the equation and then add* $2n\pi$ *to each.*

For example: Consider the equation $\sin\theta = \frac{1}{2}$, *then* $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. *Hence required solutions are* $\theta = 2n\pi + \frac{\pi}{6},\ 2n\pi + \frac{5\pi}{6}$.

2.9 Important Points to be Taken in Case of While Solving Trigonometrical Equations

(1) Check the validity of the given equation, *e.g.*, $2\sin\theta - \cos\theta = 4$ can never be true for any θ as the value $(2\sin\theta - \cos\theta)$ can never exceeds $\sqrt{2^2 + (-1)^2} = \sqrt{5}$. So there is no solution to this equation.

(2) Equation involving $\sec\theta$ or $\tan\theta$ can never have a solution of the form. $(2n+1)\frac{\pi}{2}$

Similarly, equations involving $\text{cosec}\,\theta$ or $\cot\theta$ can never have a solution of the form $\theta = n\pi$. The corresponding functions are undefined at these values of θ.

(3) If while solving an equation we have to square it, then the roots found after squaring must be checked whether they satisfy the original equation or not, *e.g.*, Let $x = 3$. Squaring, we get $x^2 = 9$ $\therefore$ $x = 3$ and -3 but $x = -3$ does not satisfy the original equation $x = 3$. *e.g.*, $\sin x + \cos x = 1$

Square both sides, we get $1 + \sin 2x = 1$ $\qquad$ $\therefore$ $\sin 2x = 0$

$\therefore$ $2x = n\pi$ $\quad$ or $\quad$ $x = \frac{n\pi}{2}$, $n \in I$

$\therefore$ Roots are $......, \frac{-3\pi}{2}, \frac{-2\pi}{2}, \frac{-\pi}{2}, 0, \frac{\pi}{2}, \frac{2\pi}{2}, \frac{3\pi}{2}, ...$

We find that 0 and $\pi/2$ are roots but π and $3\pi/2$ do not satisfy the given equation as it leads to $-1 = 1$

Similarly 0 and $\frac{-3\pi}{2}$ are roots but $-\frac{\pi}{2}$ and $-\pi$ are not roots as it will lead to $-1 = 1$.

As stated above, because of squaring we are solving the equations $\sin x + \cos x = 1$ and $\sin x + \cos x = -1$ both. The rejected roots are for $\sin x + \cos x = -1$.

(4) Do not cancel common factors involving the unknown angle on L.H.S. and R.H.S. because it may delete some solutions. *e.g.*, In the equation $\sin\theta(2\cos\theta - 1) = \sin\theta\cos^2\theta$ if we cancel $\sin\theta$ on

both sides we get $\cos^2\theta - 2\cos\theta + 1 = 0 \Rightarrow (\cos\theta - 1)^2 = 0 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 2n\pi$. But $\theta = n\pi$ also satisfies the equation because it makes $\sin\theta = 0$. So, the complete solution is $\theta = n\pi, n \in Z$.

(5) Any value of x which makes both R.H.S. and L.H.S. equal will be a root but the value of x for which $\infty = \infty$ will not be a solution as it is an indeterminate form.

Hence, $\cos x \neq 0$ for those equations which involve $\tan x$ and $\sec x$ whereas $\sin x \neq 0$ for those which involve $\cot x$ and $\text{cosec } x$.

Also exponential function is always $+ve$ and $\log_a x$ is defined if $x > 0$, $x \neq 0$ and $a > 0, a \neq 1$ $\sqrt{f(x)} = +ve$ always and not $\pm$ *i.e.* $\sqrt{(\tan^2 x)} = \tan x$ and not $\pm \tan x$.

(6) Denominator terms of the equation if present should never become zero at any stage while solving for any value of θ contained in the answer.

(7) Sometimes the equation has some limitations also *e.g.*, $\cot^2\theta + \text{cosec}^2\theta = 1$ can be true only if $\cot^2\theta = 0$ and $\text{cosec}^2\theta = 1$ simultaneously as $\text{cosec}^2\theta \geq 1$. Hence the solution is $\theta = (2n+1)\pi/2$.

(8) If $xy = xz$ then $x(y-z) = 0 \Rightarrow$ either $x = 0$ or $y = z$ or both. But $\frac{y}{x} = \frac{z}{x} \Rightarrow y = z$ only and not $x = 0$, as it will make $\infty = \infty$. Similarly if $ay = az$, then it will also imply $y = z$ only as $a \neq 0$ being a constant.

Similarly $x + y = x + z \Rightarrow y = z$ and $x - y = x - z \Rightarrow y = z$. Here we do not take $x = 0$ as in the above because x is an additive factor and not multiplicative factor.

When $\cos\theta = 0$, then $\sin\theta = 1$ or -1. We have to verify which value of $\sin\theta$ is to be chosen which satisfies the equation. $\cos\theta = 0 \Rightarrow \theta = \left(n + \frac{1}{2}\right)\pi$.

(9)Student are advised to check whether all the roots obtained by them, satisfy the equation and lie in the domain of the variable of the given equation.

2.10 Miscellaneous Examples

Example: 35 The equation $3\sin^2 x + 10\cos x - 6 = 0$ is satisfied if

(a) $x = n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$ (b) $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$ (c) $x = n\pi \pm \cos^{-1}\left(\frac{1}{6}\right)$ (d) $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{6}\right)$

Solution: (b) $3\sin^2 x + 10\cos x - 6 = 0 \Rightarrow 3(1 - \cos^2 x) + 10\cos x - 6 = 0$

on solving, $(\cos x - 3)(3\cos x - 1) = 0$. Either $\cos x = 3$ (which is not possible) or $\cos x = \frac{1}{3}$ $\Rightarrow$ $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right)$.

Example: 36 If the solutions for θ of $\cos p\theta + \cos q\theta = 0, p > 0, q > 0$ are in A.P., then the numerically smallest common difference of A.P. is **[Kerala (Engg.) 2001]**

(a) $\frac{\pi}{p+q}$ (b) $\frac{2\pi}{p+q}$ (c) $\frac{\pi}{2(p+q)}$ (d) $\frac{1}{p+q}$

Solution: (b) Given, $\cos p\theta = -\cos q\theta = \cos(\pi + q\theta) \Rightarrow p\theta = 2n\pi \pm (\pi + q\theta), n \in I \Rightarrow \theta = \frac{(2n+1)\pi}{p-q}$ or $\frac{(2n-1)\pi}{p+q}$, $n \in I$. Both the solutions form an A.P. $\theta = \frac{(2n+1)\pi}{p-q}$ gives us an A.P. with common difference $= \frac{2\pi}{p-q}$ and $\theta = \frac{(2n-1)\pi}{p+q}$ gives us an A.P. with common difference = $\frac{2\pi}{p+q}$. Certainly, $\frac{2\pi}{p+q} < \left|\frac{2\pi}{p-q}\right|$.

Example: 37 The set of values of x for which the expression $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x}$ =1 is **[MP PET 1992; UPSEAT 1993, 2002]**

(a) ϕ

(b) $\frac{\pi}{4}$

(c) $\left\{n\pi + \frac{\pi}{4} : n = 1,2,3...\right\}$

(d) $\left\{2n\pi + \frac{\pi}{4} : n = 1,2,3...\right\}$

Solution: (a) $\tan(3x - 2x) = \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}$. But this value does not satisfy the given equation.

Example: 38 If $\cos\theta + \cos 7\theta + \cos 3\theta + \cos 5\theta = 0$, then $\theta =$ **[ISM Dhanbad 1972]**

(a) $\frac{n\pi}{4}$ (b) $\frac{n\pi}{2}$ (c) $\frac{n\pi}{8}$ (d) None of these

Solution: (c) Combining θ and $7\theta, 3\theta$ and 5θ, we get $2\cos 4\theta(\cos 3\theta + \cos\theta) = 0$

$\therefore\ 4\cos 4\theta.\cos 2\theta.\cos\theta = 0 \Rightarrow 4\frac{1}{2^3 \sin\theta}(\sin 2^3\theta) = 0 \Rightarrow \sin 8\theta = 0$. Hence $\theta = \frac{n\pi}{8}$

Example: 39 If $\tan(\cot x) = \cot(\tan x)$, then $\sin 2x$ equal to

(a) $(2n+1)\frac{\pi}{4}$ (b) $\frac{4}{(2n+1)\pi}$ (c) $4\pi(2n+1)$ (d) None of these

Solution: (b) $\tan(\cot x) = \cot(\tan x) \Rightarrow \tan(\cot x) = \tan\left(\frac{\pi}{2} - \tan x\right)$

$\cot x = n\pi + \frac{\pi}{2} - \tan x$

$\tan x + \cot x = \frac{(2n+1)\pi}{2} \Rightarrow \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{(2n+1)\pi}{2}$

$\frac{1}{\sin x.\cos x} = \frac{(2n+1)\pi}{2} \Rightarrow \sin 2x = \frac{4}{(2n+1)\pi}$.

Example: 40 The sum of all solutions of the equation $\cos x.\cos\left(\frac{\pi}{3} + x\right).\cos\left(\frac{\pi}{3} - x\right) = \frac{1}{4}$, $x \in [0, 6\pi]$ is

(a) 15π (b) 30π (c) $\frac{110\pi}{3}$ (d) None of these

Solution: (b) Here, $\cos x\left(\frac{1}{4}\cos^2 x - \frac{3}{4}\sin^2 x\right) = \frac{1}{4}$ or $\frac{\cos x}{4}(4\cos^2 x - 3) = \frac{1}{4}$ or $\cos 3x = 1$

$\Rightarrow 3x = 2n\pi \Rightarrow x = \frac{2n\pi}{3}$, where $n = 0,1,2,3,4,5,6,7,8,9$; $\therefore$ The required sum $= \frac{2\pi}{3}\sum_{n=0}^{9} n = 30\pi$.

Example: 41 The equation $\sin x + \sin y + \sin z = -3$ for $0 \le x \le 2\pi, 0 \le y \le 2\pi, 0 \le z \le 2\pi$, has **[Orissa JEE 2003]**

(a) One solution

(b) Two sets of solution

(c) Four sets of solution (d) No solution

Solution: (a) Given, $\sin x + \sin y + \sin z = -3$ is satisfied only when $x = y = z = \frac{3\pi}{2}$; for $x, y, z, \in [0, 2\pi]$.

Example: 42 The solution set of $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$ in the interval $[0, 2\pi]$ is **[EAMCET 2003]**

(a) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ (b) $\left\{\frac{\pi}{3}, \pi\right\}$ (c) $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$ (d) $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

Solution: (c) $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$

$\cos\theta = \frac{-5}{4}$ which is not possible

$\therefore\ 2\cos\theta + 1 = 0$ or $\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. Solution set is $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} \in [0, 2\pi]$

Example: 43 The equation $3\cos x + 4\sin x = 6$ has **[Orissa JEE 2002]**

(a) Finite solution (b) Infinite solution (c) One solution (d) No solution

Solution: (d) $3\cos x + 4\sin x = 6$

$\Rightarrow \frac{3}{5}\cos x + \frac{4}{5}\sin x = \frac{6}{5} \Rightarrow \cos(x - \theta) = \frac{6}{5}$ $\left[\therefore \theta = \cos^{-1}\left(\frac{3}{5}\right)\right]$

So that equation has no solution.

Example: 44 The equation $\sin x + \cos x = 2$ has **[EAMCET 1986; MP PET 1998]**

(a) One solution (b) Two solution

(c) Infinite number of solution (d) No solution

Solution: (d) No solution as $|\sin x| \le 1, |\cos x| \le 1$ and both of them do not attain their maximum value for the same angle.

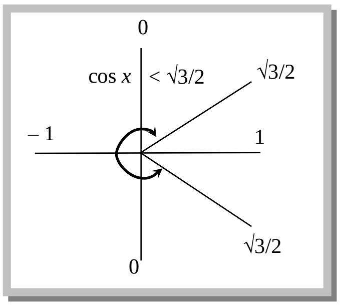
Trick: Maximum value of $\sin x + \cos x = \sqrt{1^2 + 1^2} = \sqrt{2}$, Hence there is no x satisfying this equation.

Example: 45 If $2\cos x < \sqrt{3}$ and $x \in [-\pi, \pi]$ then the solution set for x is

(a) $\left[-\pi, \frac{-\pi}{6}\right) \cup \left(\frac{\pi}{6}, \pi\right]$ (b) $\frac{-\pi}{6}, \frac{\pi}{6}$

(c) $\left[-\pi, \frac{-\pi}{6}\right] \cup \left[\frac{\pi}{6}, \pi\right]$ (d) None of these

Solution: (a) Here, $\cos x < \frac{\sqrt{3}}{2}$. The value scheme for this is shown below.



From the figure,

$-\pi \le x < \frac{-\pi}{6}$ or $\frac{\pi}{6} < x \le \pi$

$\therefore\ x \in \left[-\pi, \frac{-\pi}{6}\right) \cup \left(\frac{\pi}{6}, \pi\right]$.

Example: 46 The number of pairs (x, y) satisfying the equations $\sin x + \sin y = \sin(x + y)$ and $|x| + |y| = 1$ is

(a) 2 (b) 4 (c) 6 (d) ∞

Solution: (c) The first equation can be written as, $2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x+y)$

$\therefore$ Either $\sin\frac{1}{2}(x+y)=0$ or $\sin\frac{1}{2}x=0$ or $\sin\frac{1}{2}y=0$

$\Rightarrow x+y=0$, or $x=0$ or $y=0$. As $|x| + |y| = 1$, therefore when $x+y=0$, we have to reject $x+y=1$, or $x+y=-1$ and solve it with $x-y=1$ or $x-y=-1$ which gives $\left(\frac{1}{2},\frac{-1}{2}\right)$ or $\left(\frac{-1}{2},\frac{1}{2}\right)$ as the possible solution. Again solving with $x=0$, we get $(0,\pm 1)$ and solving with $y=0$, we get $(\pm 1,0)$ as the other solution. Thus we have six pairs of solution for x and y.

Example: 47 If $\cos\theta = \frac{-1}{2}$ and $0° < \theta < 360°$, then the values of θ are

(a) $120°$ and $300°$ (b) $60°$ and $120°$ (c) $120°$ and $240°$ (d) $60°$ and $240°$

Solution: (c) Given, $\cos\theta = -\frac{1}{2}$ and $0° < \theta < 360°$. We know that $\cos 60° = \frac{1}{2}$ and $\cos(180° - 60°) = -\cos 60° = \frac{-1}{2}$ or $\cos 120° = -\frac{1}{2}$. Similarly $\cos(180° + 60°) = -\cos 60° = -\frac{1}{2}$ or $\cos 240° = \frac{-1}{2}$.

Therefore $\theta = 120°$ and $240°$.

Example: 48 If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then the value of $\cos\left(\theta - \frac{\pi}{4}\right) =$

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{4\sqrt{2}}$

Solution: (a) $\tan(\pi\cos\theta) = \tan(\frac{\pi}{2} - \pi\sin\theta)$

$\therefore \sin\theta + \cos\theta = \frac{1}{2} \Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$.

Example: 49 The only value of x for which $2^{\sin x} + 2^{\cos x} > 2^{1-(1/\sqrt{2})}$ hold is

(a) $\frac{5\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{2}$ (d) All values of x.

Solution: (a) Since A.M. $\geq$ G.M.

$\frac{1}{2}(2^{\sin x} + 2^{\cos x}) \geq \sqrt{2^{\sin x}.2^{\cos x}} \Rightarrow 2^{\sin x} + 2^{\cos x} \geq 2.2^{\frac{\sin x + \cos x}{2}} \Rightarrow 2^{\sin x} + 2^{\cos x} \geq 2^{1+\frac{\sin x+\cos x}{2}}$

And, we know that $\sin x + \cos x \geq -\sqrt{2}$

$\therefore 2^{\sin x} + 2^{\cos x} > 2^{1-(1/\sqrt{2})}$ for $x = \frac{5\pi}{4}$.

2.11 Periodic Functions

A function $f(x)$ is called periodic function if there exists a least positive real number T such that $f(x+T) = f(x)$. T is called the period (or fundamental period) of function $f(x)$. Obviously, if T is the period of $f(x)$, then $f(x) = f(x+T) = f(x+2T) = f(x+3T) = ...$

(i) If $f_1(x)$ and $f_2(x)$ are two periodic functions of x having the same period T, then the function $af_1(x) + bf_2(x)$ where a and b are any numbers, is also a periodic function having the same period T.

(ii) If T is the period of the periodic function $f(x)$, then the function $f(ax+b)$, where $a(>0)$ and b are any numbers is also a periodic function with period equal to T/a.

(iii) If T_1 and T_2 are the periods of periodic functions $f_1(x)$ and $f_2(x)$ respectively, then the function $af_1(x)+bf_2(x)$, where a and b are any numbers is also periodic and its period is T which is the L.C.M. of T_1 and T_2 *i.e.* T is the least positive number which is divisible by T_1 and T_2.

All trigonometric functions are periodic. The period of trigonometric function $\sin x, \cos x, \sec x$ and $\operatorname{cosec} x$ is 2π because $\sin(x+2\pi)=\sin x, \cos(x+2\pi)=\cos x$ etc.

The period of $\tan x$ and $\cot x$ is π because $\tan(x+\pi)=\tan x$ and $\cot(x+\pi)=\cot x$

The period of the function which are of the type: $\sin ax, \cos(ax+b); b\cos ax$ is $\dfrac{2\pi}{a}$

The period of $\tan ax$ and $\cot ax$ is $\dfrac{\pi}{|a|}$. Here |a| is taken so as the value of the period is positive real number.

Some functions with their periods

Function	Period
$\sin(ax+b), \cos(ax+b), \sec(ax+b), \operatorname{cosec}(ax+b)$	$2\pi/a$
$\tan(ax+b), \cot(ax+b)$	π/a
$\|\sin(ax+b)\|, \|\cos(ax+b)\|, \|\sec(ax+b)\|, \|\operatorname{cosec}(ax+b)\|$	π/a
$\|\tan(ax+b)\|, \|\cot(ax+b)\|$	π/a

Example: 50 Period of $\sin^2 x$ is **[UPSEAT 2002; AIEEE 2002]**

(a) π (b) 2π (c) $\dfrac{\pi}{2}$ (d) None of these

Solution: (a) $\sin^2 x=\dfrac{1-\cos 2x}{2}$ $\Rightarrow$ Period $=\dfrac{2\pi}{2}=\pi.$

Example: 51 The period of the function $y=\sin 2x$ is **[Kerala(Engg.) 2003]**

(a) 2π (b) π (c) $\dfrac{\pi}{2}$ (d) 4π

Solution: (a) Period of $\sin(ax+b)=\dfrac{2\pi}{|a|}$

$\therefore$ Period of $\sin 2x=\dfrac{2\pi}{|2|}=\pi.$

Example: 52 The period of the function $f(\theta)=\sin\dfrac{\theta}{3}+\cos\dfrac{\theta}{2}$ is **[EAMCET 2001]**

(a) 3π (b) 6π (c) 9π (d) 12π

Solution: (d) Period of $\sin\left(\dfrac{\theta}{3}\right)=6\pi$ and period of $\cos\left(\dfrac{\theta}{2}\right)=4\pi$

L.C.M. of 6π and $4\pi=12\pi$

Example: 53 The function $f(x) = \sin\frac{\pi x}{2} + 2\cos\frac{\pi x}{3} - \tan\frac{\pi x}{4}$ is periodic with period **[Rajasthan PET 2001]**

(a) 6 (b) 3 (c) 4 (d) 12

Solution: (d) Period of $\sin\frac{\pi x}{2} = \frac{2\pi}{\pi/2} = 4$

Period of $\cos\frac{\pi x}{3} = \frac{2\pi}{\pi/3} = 6$ and period of $\tan\frac{\pi x}{4} = \frac{\pi}{\pi/4} = 4$

$\therefore$ Period of $f(x) =$ L.C.M. of (4, 6, 4)=12.

Example: 54 If the period of the function $f(x) = \sin\left(\frac{x}{n}\right)$ is 4π, then n is equal to **[Pb. CET 2000]**

(a) 1 (b) 4 (c) 8 (d) 2

Solution: (d) $\sin\left(\frac{x}{n}\right) = \sin\left(2\pi + \frac{x}{n}\right) = \sin\left[\frac{1}{n}(2n\pi + x)\right]$ $\Rightarrow$ Period of the function $\sin\left(\frac{x}{n}\right)$ is $2n\pi$. $\Rightarrow$ $2n\pi = 4\pi \Rightarrow n = 2$.

Example: 55 The period of $\sin^4 x + \cos^4 x$ is

(a) $\frac{\pi}{2}$ (b) π (c) 2π (d) $\frac{3\pi}{2}$

Solution: (a) $(\sin^2 x)^2 + (\cos^2 x)^2 = 1 - 2\sin^2 x\cos^2 x = 1 - \frac{1}{4}2\sin^2 2x = 1 - \left(\frac{1-\cos 4x}{4}\right) = \frac{3}{4} + \frac{1}{4}\cos 4x$

Therefore period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

Trick: $f(x) = \sin^4 x + \cos^4 x \Rightarrow f\left(\frac{\pi}{2} + x\right) = \sin^4\left(\frac{\pi}{2} + x\right) + \cos^4\left(\frac{\pi}{2} + x\right) \Rightarrow f\left(\frac{\pi}{2} + x\right) = \cos^4 x + \sin^4 x = f(x)$

Hence the period is $\frac{\pi}{2}$.

Example: 56 Period of $\sin\theta - \sqrt{3}\cos\theta$ is **[MP PET 1990]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π

Solution: (d) $\sin\theta - \sqrt{3}\cos\theta = 2\left(\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}.\cos\theta\right) = 2\sin\left(\theta - \frac{\pi}{3}\right)$

Hence period $= 2\pi$.

Example: 57 Period of $|\sin 2x|$ is **[MP PET 1989]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π

Solution: (b) Period of $\sin 2x = \pi$ and period of $|\sin 2x| = \frac{\pi}{2}$.

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